

# Berry Phase and Pseudospin Winding Number in Bilayer Graphene

Cheol-Hwan Park<sup>1,2\*</sup> and Nicola Marzari<sup>2,3</sup>

<sup>1</sup>*Department of Materials Science and Engineering,  
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

<sup>2</sup>*Department of Materials, University of Oxford, Oxford OX1 3PH, UK*

<sup>3</sup>*Theory and Simulation of Materials, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland*

(Dated: January 20, 2013)

Ever since the novel quantum Hall effect in bilayer graphene was discovered, and explained by a Berry phase of  $2\pi$  [K. S. Novoselov *et al.*, “Unconventional quantum Hall effect and Berry’s phase of  $2\pi$  in bilayer graphene”, *Nature Phys.* **2**, 177 (2006)], it has been widely accepted that the low-energy electronic wavefunction in this system is described by a non-trivial Berry phase of  $2\pi$ , different from the zero phase of a conventional two-dimensional electron gas. Here, we show that (i) the relevant Berry phase for bilayer graphene is not different from that for a conventional two-dimensional electron gas (as expected, given that Berry phase is only meaningful modulo  $2\pi$ ) and that (ii) what is actually observed in the quantum Hall measurements is not the absolute value of the Berry phase but the pseudospin winding number.

## I. INTRODUCTION

A conventional two-dimensional electron gas (2DEG) under a high magnetic field and low temperature shows a quantized Hall conductance  $\sigma_{xy} = n e^2/h$  per spin degree of freedom where  $n$  is an integer,  $e$  the charge of an electron, and  $h$  Planck’s constant [1]. For graphene, it has been predicted that the Hall conductance per spin and valley degrees of freedom is  $\sigma_{xy} = (n + 1/2) e^2/h$  [2–4], and this prediction has then been proven by experiments performed on mechanically exfoliated samples [5, 6]. Such half-integer quantum Hall effect is the most decisive evidence that the sample is actually a single atomic layer of carbon atoms effectively decoupled from the substrate, and is a direct manifestation of a non-trivial Berry phase of  $\pi$  in the electron wavefunction [5, 6]. (This half-integer quantum Hall effect is different from the fractional quantum Hall effect in graphene [7–10].) Half-integer quantum Hall effect has been observed in graphene epitaxially grown on the carbon-rich surface [11] and the silicon-rich surface [12–14] of silicon carbide or in graphene grown by

chemical vapor deposition [15].

When a second layer is added — thus forming bilayer graphene [Fig. 1(a)] — the electronic structure changes dramatically. The dispersion for low-energy quasiparticles in graphene is linear [Fig. 2(a)], whereas in bilayer graphene it becomes quadratic [Fig. 2(b)]. The quantum Hall conductance in bilayer graphene has been theoretically predicted [16], and experimentally verified (again by using mechanically exfoliated samples) [17] to be  $\sigma_{xy} = n e^2/h$  per spin and valley degrees of freedom. This is very similar to the case of a conventional 2DEG, but, very significantly, with no plateau at  $n = 0$ . This novel quantum Hall effect in bilayer graphene has been explained by the appearance of a non-trivial Berry phase of  $2\pi$  in the electronic wavefunction [16, 17], different from the case of a conventional 2DEG. These pioneering studies [16, 17] have significantly influenced the community working on the novel physics of graphene nanostructures, and the concept of a non-trivial Berry phase in the electronic wavefunction of bilayer graphene has been commonly accepted thereafter.

In this paper, we show that the Berry phase in the electronic wavefunction of bilayer graphene is equivalent to that of a conventional 2DEG (i. e., Berry phase of  $2\pi$ ,  $4\pi$ , etc., are *not* non-trivial but are all equivalent to 0) by adopting simple arguments on the gauge freedom of electronic wavefunctions. We then reinterpret the original experimental results [17]: although the quantum Hall effect in bilayer graphene is qualitatively different from that in a conventional 2DEG, what we need for an explanation of that novel phenomenon is not a non-trivial Berry phase of  $2\pi$  but the concept of *pseudospin winding number*, i. e., how many times the pseudospin vector — the vector representing the relative phase on the two different sublattices of carbon atoms [18] [see lower panels of Figs. 2(a) and 2(b), and Eqs. (7) and (12)] — rotates when the electronic wavevector undergoes one full rotation around the Dirac point [Figs. 1(b) and 3]. We note

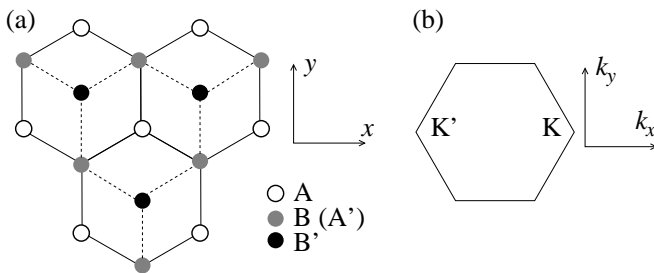


FIG. 1: (a) Schematic representation of graphene and bilayer graphene. For graphene, the carbon atoms belong only to the two A and B sublattices. (b) The Brillouin zone of graphene and bilayer graphene. The positions of the K and K’ points are  $(\frac{4\pi}{3a}, 0)$  and  $(-\frac{4\pi}{3a}, 0)$ , respectively, where  $a$  is the lattice parameter.

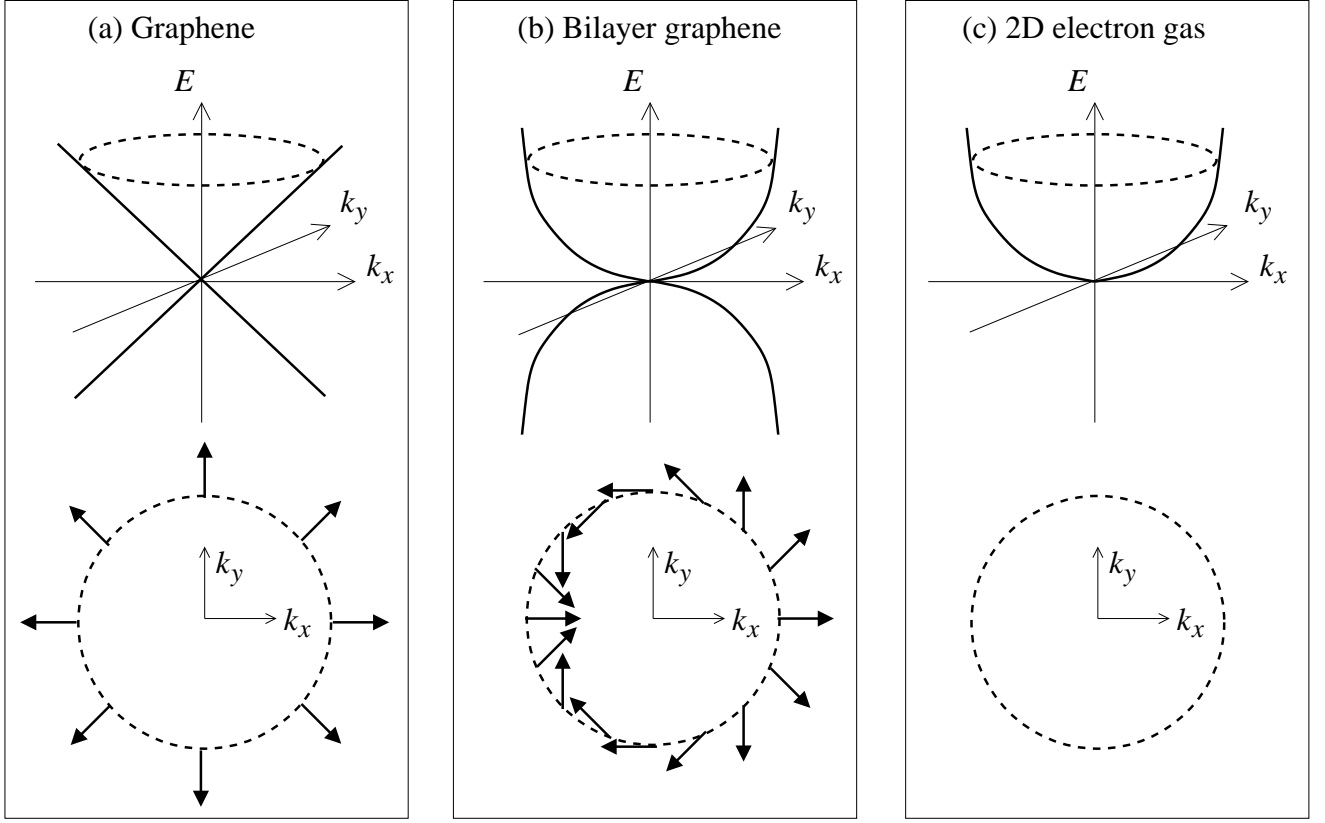


FIG. 2: (a) Upper panel: Energy dispersions for low-energy electrons in graphene. Lower panel: Pseudospin distribution for electronic eigenstates in graphene on an equi-energy contour specified by the dashed curve in the upper panel. The arrows represent the direction of pseudospin. Here, we consider the electronic states with wavevectors near the Dirac point K. (b) and (c): Similar quantities as in (a) for bilayer graphene and a conventional 2DEG, respectively. Note that the electrons in a 2DEG have an energy minimum at the center of the Brillouin zone and do not have a pseudospin degree of freedom.

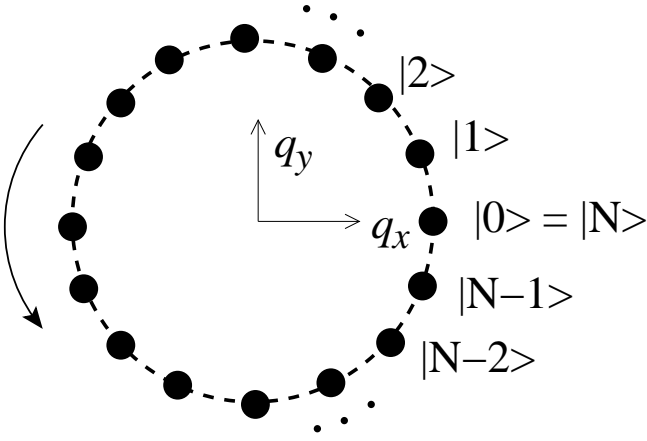


FIG. 3: Schematics of electronic eigenstates in graphene or multi-layer graphene on an equi-energy contour with a wavevector around the Dirac point K. We define  $\mathbf{q} = \mathbf{k} - \mathbf{K}$  where  $\mathbf{k}$  is the 2D Bloch wavevector. If we start from the state  $|0\rangle$ , we end up with the state  $|N\rangle$  after a complete counter-clockwise rotation along the contour ( $|N\rangle = |0\rangle$ ). Here, the evolving parameter is  $\mathbf{q}$ .

that this pseudospin winding number is exactly the same quantity as the ‘degree of chirality’  $J$  of previous studies [16, 17]. On the other hand, the term ‘chirality’ – in connection with neutrino physics – has been used to refer to the inner product between the two-dimensional (2D) Bloch wavevector from the Dirac point and the pseudospin vector [19]; hence, for graphene, the chiralities for electronic states with wavevector close to the K point in the upper band and in the lower band would be  $+1$  and  $-1$ , respectively, whereas the degree of chirality is  $+1$  regardless of the band index. To avoid this confusion, we suggest the term ‘pseudospin winding number’ here. Also, pseudospin winding number has been defined in the form of a more general topological invariant [20–22].

## II. BERRY PHASE

For electrons in a periodic system, the Berry phase is a phase acquired by the wavefunction over the course of a cyclic evolution of the Hamiltonian – such concept has had a fundamental role in developing the modern theory of polarization [23, 24] and of magnetization [25, 26].

The evolving parameter that we consider here is the 2D Bloch wavevector  $\mathbf{k} = (k_x, k_y)$ . The Berry phase  $\Gamma$  described in the wavevector evolution of Fig. 3 is given by

$$\Gamma = -i \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} \log \langle j | j+1 \rangle, \quad (1)$$

where

$$|N\rangle = |0\rangle. \quad (2)$$

Here,  $\mathbf{k}$  is determined by the state index  $j$ .

We first show that Berry phase [defined in Eq. (1)] is in general meaningful only modulo  $2\pi$ . After multiplying each of the wavefunctions by a position-independent overall phase factor  $e^{i\theta_j}$ , we obtain another set of wavefunctions which satisfies the equation of motion

$$|j\rangle' = e^{i\theta_j} |j\rangle. \quad (3)$$

In order to have a well-defined set of wavefunctions for Berry phase evaluation [Eq. (2)], we have to impose

$$\theta_N - \theta_0 = 2\pi m \quad (4)$$

where  $m$  is an arbitrary integer. Now using Eqs. (1), (3), and (4), the Berry phase  $\Gamma'$  for the new set of wavefunctions [Eq. (3)] is

$$\begin{aligned} \Gamma' &= -i \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} \log \langle j | j+1 \rangle' \\ &= \Gamma + \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} (\theta_{j+1} - \theta_j) \\ &= \Gamma + 2\pi m, \end{aligned} \quad (5)$$

i.e., Berry phase can be defined modulo  $2\pi$  only. In the following, we paraphrase this general argument using the language of graphene.

For graphene, let us consider the case where  $\mathbf{k}$  is very close to the Dirac point K [Fig. 1(b)], and define  $\mathbf{q} = \mathbf{k} - \mathbf{K}$  ( $|\mathbf{q}| \ll |\mathbf{K}|$ ). Then, if we use a basis set composed of the Bloch sums of  $p_z$  orbitals localized on the two sublattices A and B [Fig. 1(a)], the effective Hamiltonian reads [18]

$$H_{\text{mono}}(\mathbf{q}) = \hbar v_0 q \begin{pmatrix} 0 & \exp(-i\theta_{\mathbf{q}}) \\ \exp(i\theta_{\mathbf{q}}) & 0 \end{pmatrix}, \quad (6)$$

where  $v_0$  is the band velocity and  $\theta_{\mathbf{q}}$  the angle between  $\mathbf{q}$  and the  $+k_x$  direction. The energy eigenvalue and wavefunction of Eq. (6) are given by  $E_{s\mathbf{q}}^{\text{mono}} = \hbar v_0 s |\mathbf{q}|$  and

$$|\psi_{s\mathbf{q}}^{\text{mono}}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ s e^{i\theta_{\mathbf{q}}} \end{pmatrix}, \quad (7)$$

respectively, where  $s = \pm 1$  is the band index. As shown before [27], the Berry phase  $\Gamma_{\text{mono}}$  of the electron wavefunction in graphene when the Bloch wavevector undergoes a full rotation around the Dirac point (Fig. 3) can be obtained from Eq. (1):

$$\begin{aligned} \Gamma_{\text{mono}} &= -i \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} \log \left( \frac{1 + \exp[i(\theta_{j+1} - \theta_j)]}{2} \right) \\ &= -i \int_0^{2\pi} d\theta \frac{i}{2} \\ &= \pi, \end{aligned} \quad (8)$$

where we have substituted  $\mathbf{q}$  by the corresponding state index  $j$  ( $j = 0, 1, 2, \dots, N-1$ ) as shown in Fig. 3.

Now we show that Eq. (1) is not affected by an arbitrary phase factor of each electronic wavefunction. Suppose that we use a new form for an electron wavefunction in graphene

$$\begin{aligned} |\psi_{s,j}^{\text{mono}}\rangle' &= e^{-i\theta_j/2} |\psi_{s,j}^{\text{mono}}\rangle \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta_j/2} \\ s e^{i\theta_j/2} \end{pmatrix}, \end{aligned} \quad (9)$$

for  $j = 0, 1, \dots, N-1$ , and of course  $|\psi_{s,N}^{\text{mono}}\rangle' = |\psi_{s,0}^{\text{mono}}\rangle'$  (Fig. 3). Then the Berry phase  $\Gamma'_{\text{mono}}$  is given by

$$\begin{aligned} \Gamma'_{\text{mono}} &= -i \lim_{N \rightarrow \infty} \{ \log \langle \psi_{s,N-1}^{\text{mono}} | \psi_{s,0}^{\text{mono}} \rangle' \\ &\quad + \sum_{j=0}^{N-2} \log \left( \frac{e^{-i(\theta_{j+1} - \theta_j)/2} + e^{i(\theta_{j+1} - \theta_j)/2}}{2} \right) \} \\ &= -i \left\{ i\pi + \int_0^{2\pi} d\theta 0 \right\} \\ &= \pi, \end{aligned} \quad (10)$$

i.e., we obtain the same result as in Eq. (8). The physical reason for this invariance of the Berry phase with respect to an arbitrary gauge phase of each electronic state evaluated from Eq. (1) is that each state appears twice (once as a bra state and once as a ket state), thus cancelling out any arbitrary phase in each wavefunction. Since the Berry phase obtained from Eq. (1) is *not* affected by any arbitrary phase of each state  $|j\rangle$ , we can choose for convenience a gauge such that the state varies smoothly over the course of parameter evolution.

Similarly, for bilayer graphene, if we use a basis set composed of Bloch sums of localized Wannier-like  $p_z$  orbitals on each of the two sublattices A and B' [Fig. 1(a)] [28], and consider only the nearest-neighbor intra-layer hopping and vertical interlayer hopping (characterized by the hopping integral  $t_{\perp}$ ) processes, the effective Hamiltonian of low-energy electronic states in bilayer graphene becomes [17]

$$H_{\text{bi}}(\mathbf{q}) = -\frac{\hbar^2 q^2}{2m^*} \begin{pmatrix} 0 & \exp(-2i\theta_{\mathbf{q}}) \\ \exp(2i\theta_{\mathbf{q}}) & 0 \end{pmatrix}, \quad (11)$$

where  $m^* = 3a^2 t_\perp / 8v_0^2$  is the effective mass and  $a$  the lattice parameter. The energy eigenvalue and wavefunction of Eq. (11) are given by  $E_{s\mathbf{q}}^{\text{bi}} = s\hbar^2 q^2 / 2m^*$  and

$$|\psi_{s\mathbf{q}}^{\text{bi}}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -s e^{2i\theta_{\mathbf{q}}} \end{pmatrix}, \quad (12)$$

respectively ( $s = \pm 1$ ). The Berry phase  $\Gamma_{\text{bi}}$  of the electron wavefunction in bilayer graphene can again be obtained from Eq. (1):

$$\begin{aligned} \Gamma_{\text{bi}} &= -i \lim_{N \rightarrow \infty} \sum_{j=0}^{N-1} \log \left( \frac{1 + \exp[2i(\theta_{j+1} - \theta_j)]}{2} \right) \\ &= -i \int_0^{2\pi} d\theta i \\ &= 2\pi, \end{aligned} \quad (13)$$

which is the result of previous studies [16, 17].

Now we consider another form for the wavefunction

$$|\psi_{s\mathbf{q}}^{\text{bi}}\rangle' = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\theta_{\mathbf{q}}} \\ -s e^{i\theta_{\mathbf{q}}} \end{pmatrix}, \quad (14)$$

whose only difference from the original one is the position-independent overall phase ( $|\psi_{s\mathbf{q}}^{\text{bi}}\rangle' = e^{-i\theta_{\mathbf{q}}} |\psi_{s\mathbf{q}}^{\text{bi}}\rangle$ ). Because the two wavefunctions are different only in the overall coefficient, they both are perfectly good solutions of the effective Hamiltonian of bilayer graphene [Eq. (11)]. Moreover, both the original [Eq. (12)] and new [Eq. (14)] wavefunctions satisfy the gauge condition that the wavefunction varies smoothly in the course of cyclic evolution; hence, in the evaluation of the Berry phase, one does not have to consider any branch-cut as we did in the derivation of Eq. (10). The Berry phase  $\Gamma'_{\text{bi}}$  for  $|\psi_{s\mathbf{q}}^{\text{bi}}\rangle'$  evaluated from Eq. (1) is

$$\Gamma'_{\text{bi}} = 0; \quad (15)$$

in other words, *the Berry phase of the electronic wavefunction of bilayer graphene is the same as that of a conventional 2DEG.*

In fact, the Berry phase of an electronic wavefunction in bilayer graphene evaluated from Eq. (1) is  $2m\pi$ , with  $m$  an arbitrary integer. Or, in other words, any Berry phase of  $2m\pi$  is equivalent to a trivial Berry phase of 0. [Similarly, this Berry phase for graphene obtained from Eq. (1) is  $(2m+1)\pi$ , with  $m$  an arbitrary integer.] This statement is related also to the well known application of Berry-phase physics to the modern theory of polarization [23, 24], according to which the electrical polarization in a periodic system can only be determined modulo  $e\mathbf{R}/\Omega$ , where  $\mathbf{R}$  is a lattice vector and  $\Omega$  the unit cell volume.

Our claim that the Berry phase in bilayer graphene is *not* non-trivial and is equivalent to that of a conventional 2DEG remains valid even if we consider a more

complex model Hamiltonian. If, for example, second-nearest-neighbor inter-layer hopping processes are considered in the tight-binding model of bilayer graphene, the low-energy electronic states can be described by four Dirac cones [16]. It was suggested that the central Dirac cone contributes  $-\pi$  to the Berry phase and each of the three satellite Dirac cones contribute  $\pi$  to the Berry phase and hence the total “non-trivial” Berry phase is  $-\pi + 3 \times \pi = 2\pi$  [16]. According to our discussion, this view is not correct, again because a Berry phase of  $-\pi$  is equivalent to that of  $\pi$ , etc. In particular, starting from the wavefunctions used in Ref. [16], one can define a new set of wavefunctions with additional four phase factors, e.g.,  $e^{-i\theta_{\mathbf{q}}}$ , defined around each of the four Dirac cones in a way similar to Eq. (14). Then, the Berry phase around the  $i^{\text{th}}$  Dirac cone ( $i = 1, 2, 3, 4$ ) obtained by evaluating Eq. (1) would be  $(2m_i+1)\pi$  for an arbitrary integer  $m_i$ . Thus, the Berry phase for bilayer graphene is of the form  $\Gamma_{\text{bi}} = \sum_{i=1}^4 (2m_i+1)\pi = 2m\pi$ , with  $m$  an arbitrary integer, and cannot uniquely be  $2\pi$  as suggested in previous studies [16, 17].

### III. PSEUDOSPIN WINDING NUMBER

However, these considerations do not mean that there is no qualitative difference between the electronic eigenstates in bilayer graphene and those in a conventional 2DEG. Indeed, a new phenomenon in bilayer graphene was observed [17] and explained [16]: the step size at charge neutrality point in the quantum Hall conductance  $\sigma_{xy}$  in bilayer graphene is  $2e^2/h$  per spin and valley degrees of freedom and not  $e^2/h$  as in a conventional 2DEG. We reinterpret this result as a direct manifestation of the *pseudospin winding number*, and explain this in the following.

For an electronic state whose wavefunction can be defined as a two-component spinor, as in graphene [Eq. (7)] or in bilayer graphene [Eq. (12)], a 2D pseudospin vector can be defined from the relative phase of the two components [17, 18]. Figures 2(a) and 2(b) show the pseudospin vector for electronic wavefunctions in graphene and in bilayer graphene, respectively. The pseudospin winding number  $n_w$  is then defined as the number of rotations that a pseudospin vector undergoes when the electronic wavevector rotates fully one time around the Dirac point: it is immediate to see from Figs. 2(a) and 2(b) that  $n_w$  is 1 in graphene and 2 in bilayer graphene.

The low-energy effective Hamiltonian of a graphene multilayer, with  $n$  layers and ABC stacking, is [29]

$$H_n(\mathbf{q}) \propto q^n \begin{pmatrix} 0 & \exp(-ni\theta_{\mathbf{q}}) \\ \exp(ni\theta_{\mathbf{q}}) & 0 \end{pmatrix}. \quad (16)$$

Eigenvalues and eigenfunctions of Eq. (16) are given by

$E_{s\mathbf{q}}^n \propto s q^n$  and

$$|\psi_{s\mathbf{q}}^n\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -s e^{ni\theta_{\mathbf{q}}} \end{pmatrix} \quad (17)$$

respectively ( $s = \pm 1$ ). Thus, the pseudospin winding number  $n_w$  of the eigenstate in Eq. (16) is  $n$ .

It has been shown (Supplementary Information of Ref. [17]) that a system described by Eq. (16) has a quantum Hall conductance step at the charge neutrality point of size  $n e^2/h$  per spin and valley degrees of freedom. Our interpretation is that what we learn from a measured step size  $n e^2/h$  per spin and valley degrees of freedom [17] is not the “absolute” value of the Berry phase ( $\Gamma = n\pi$ ) but the pseudospin winding number ( $n_w = n$ ); We also note that the quantum Hall conductance step, which has been correctly obtained in Ref. [17] from Eq. (16), does not need a Berry phase of  $\Gamma = n\pi$  for its explanation. In other words, the size of the quantum Hall conductance step at the charge neutrality point is connected directly to the special form of the Hamiltonian in Eq. (16), or, equivalently, to  $n$  which can be interpreted, from its wavefunction [Eq. (17)], as the pseudospin winding number and not to the absolute value of the Berry phase. (It should be stressed that our discussion of the pseudospin winding number is confined to systems whose effective Hamiltonian is given by Eq. (16).)

On the other hand, the Berry phase still determines the shift of the plateaus, with integer and half-integer quantum Hall effects for  $\Gamma = 0$  and  $\Gamma = \pi$ , respectively [5, 6, 27]. The former and the latter correspond to multi-layer graphene with even and odd number of layers, respectively [29]. It has been recently shown within a semiclassical theory that the so-called topological part of the Berry phase, which is determined by the pseudospin winding number, is responsible for this shift [30].

#### IV. CONCLUSION

In conclusion, we have shown that the Berry phase of the low-energy quasiparticle wavefunction of bilayer graphene is the same as that of a conventional two-dimensional electron gas, and that the fundamental difference between the two systems is the pseudospin winding number. Our findings have a broader implication than the cases discussed here. For example, the interpretation of the recent angle-resolved photoemission spectroscopy on bilayer graphene [31, 32] and that of the recent quantum Hall experiments on trilayer graphene [33] should be revisited in view of the present discussion.

#### ACKNOWLEDGMENTS

We thank Steven G. Louie, Alessandra Lanzara, Choonkyu Hwang, and Davide Ceresoli for fruitful dis-

cussions, and Intel Corporation for support (CHP).

---

\* Electronic address: chpark77@mit.edu

- [1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
- [2] Y. Zheng and T. Ando, Phys. Rev. B **65**, 245420 (2002).
- [3] V. P. Gusynin and S. G. Sharapov, Phys. Rev. Lett. **95**, 146801 (2005).
- [4] N. M. R. Peres, F. Guinea, and A. H. Castro Neto, Ann. Phys. **321**, 1559 (2006).
- [5] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature **438**, 197 (2005).
- [6] Y. Zhang, J. W. Tan, H. L. Stormer, and P. Kim, Nature **438**, 201 (2005).
- [7] X. Du, I. Skachko, F. Duerr, A. Luican, and E. Y. Andrei, Nature **462**, 192 (2009).
- [8] K. I. Bolotin, F. Ghahari, M. D. Shulman, H. L. Stormer, and P. Kim, Nature **462**, 196 (2009).
- [9] F. Ghahari, Y. Zhao, P. Cadden-Zimansky, K. Bolotin, and P. Kim, Phys. Rev. Lett. **106**, 046801 (2011).
- [10] C. R. Dean, A. F. Young, P. Cadden-Zimansky, L. Wang, H. Ren, K. Watanabe, T. Taniguchi, P. Kim, J. Hone, and K. L. Shepard, Nat. Phys. **7**, 693 (2011).
- [11] X. Wu, Y. Hu, M. Ruan, N. K. Madiomanana, J. Hankinson, M. Sprinkle, C. Berger, and W. A. de Heer, Appl. Phys. Lett. **95**, 223108 (2009).
- [12] T. Shen, J. J. Gu, M. Xu, Y. Q. Wu, M. L. Bolen, M. A. Capano, L. W. Engel, and P. D. Ye, Appl. Phys. Lett. **95**, 172105 (2009).
- [13] A. Tzalenchuk, S. Lara-Avila, A. Kalaboukhov, S. Paoilillo, M. Syvajarvi, R. Yakimova, O. Kazakova, J. J. B. M., V. Fal’ko, and S. Kubatkin, Nat. Nanotechnol. **5**, 186 (2010).
- [14] J. Jobst, D. Waldmann, F. Speck, R. Hirner, D. K. Maude, T. Seyller, and H. B. Weber, Phys. Rev. B **81**, 195434 (2010).
- [15] K. S. Kim, Y. Zhao, H. Jang, S. Y. Lee, J. M. Kim, K. S. Kim, J.-H. Ahn, P. Kim, J.-Y. Choi, and B. H. Hong, Nature **457**, 706 (2009).
- [16] E. McCann and V. I. Fal’ko, Phys. Rev. Lett. **96**, 086805 (2006).
- [17] K. S. Novoselov, E. McCann, S. V. Morozov, V. I. Fal’ko, M. I. Katsnelson, U. Zeitler, D. Jiang, F. Schedin, and A. K. Geim, Nat. Phys. **2**, 177 (2006).
- [18] P. R. Wallace, Phys. Rev. **71**, 622 (1947).
- [19] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, Rev. Mod. Phys. **81**, 109 (2009).
- [20] T. T. Heikkilä and G. E. Volovik, JETP Lett. **92**, 751 (2010).
- [21] G. E. Volovik, *Springer Lecture Notes in Physics* (Springer, Heidelberg, 2007), Vol. 718, pp. 31–73.
- [22] X. G. Wen and A. Zee, Phys. Rev. B **66**, 235110 (2002).
- [23] R. D. King-Smith and D. Vanderbilt, Phys. Rev. B **47**, 1651 (1993).
- [24] R. Resta, Rev. Mod. Phys. **66**, 899 (1994).
- [25] T. Thonhauser, D. Ceresoli, D. Vanderbilt, and R. Resta, Phys. Rev. Lett. **95**, 137205 (2005).
- [26] D. Ceresoli, T. Thonhauser, D. Vanderbilt, and R. Resta,

- Phys. Rev. B **74**, 024408 (2006).
- [27] T. Ando, T. Nakanishi, and R. Saito, J. Phys. Soc. Jpn **67**, 2857 (1998).
  - [28] Y.-S. Lee, M. Buongiorno Nardelli, and N. Marzari, Phys. Rev. Lett. **95**, 076804 (2005).
  - [29] H. Min and A. H. MacDonald, Phys. Rev. B **77**, 155416 (2008).
  - [30] J. N. Fuchs, F. Piechon, M. O. Goerbig, and G. Montambaux, Eur. Phys. J. B **77**, 351 (2010).
  - [31] C. Hwang, C.-H. Park, D. A. Siegel, A. V. Fedorov, S. G. Louie, and A. Lanzara, Phys. Rev. B **84**, 125422 (2011).
  - [32] Y. Liu, G. Bian, T. Miller, and T.-C. Chiang, Phys. Rev. Lett. **107**, 166803 (2011).
  - [33] L. Zhang, Y. Zhang, J. Camacho, M. Khodas, and I. Zaliznyak, Nat. Phys., published online, 25-SEP-2011, DOI:10.1038/nphys2104.